# Lecture 20

## 9.5 - Linear First Order ODEs.

A linear first order ODE is one of the form

$$y' + p(x)y = g(x). \quad (*)$$

The method we will use to solve this is the method of <u>integrating factors</u>.

<u>Def</u>: An <u>integrating factor</u> for $(*)$ is a function $\mu(x)$ such that multiplying $(*)$ by $\mu(x)$ lets us write

$$(\mu y)' = \mu y' + \mu p y = \mu g \qquad (**)$$

and integrating this lets us solve for $y$ as:

$$\int (\mu y)' dx = \int \mu g \, dx \Rightarrow \mu y = \int \mu g \, dx$$

$$\Rightarrow y = \frac{1}{\mu} \int \mu g \, dx$$

Using (**), we can derive a formula for $\mu(x)$ as well:

From (**): $(\mu y)' = \mu y' + \mu p y$

$$\Rightarrow \mu' y + \cancel{\mu y'} = \cancel{\mu y'} + \mu p y$$

$$\Rightarrow \mu' \cancel{y} = \mu p \cancel{y} \Rightarrow \mu' = \mu p$$

$$\Rightarrow \mu = e^{\int p(x)\,dx}$$

Note: $\int p(x)\,dx$ comes with a constant of integration, which we can choose to be whatever we want since we only need <u>an</u> integrating factor. We usually choose it to be zero for simplicity.

In summary, the solution to $y' + p(x)y = g(x)$ is given by

$$\boxed{y = \frac{1}{\mu}\int \mu g\,dx}$$

where $\mu$ is the integrating factor given by

$$\boxed{\mu = e^{\int p\,dx}}$$

Ex: Solve $x(\ln x)y' + y = 2\ln x$, $x > 1$. typo

Sol: First, write in form: $y' + p(x)y = g(x)$:

$$y' + \frac{1}{x\ln x}y = \frac{2}{x} \quad (x\ln x \neq 0 \text{ for } x>1)$$

Integrating factor $\mu = e^{\int \frac{1}{x\ln x}dx} = e^{\ln|\ln x|} = |\ln x| = \ln x$ $\left(\begin{matrix}\ln x>0 \\ \text{for } x>1\end{matrix}\right)$

Solution: $y = \frac{1}{\mu}\int \mu g\, dx = \frac{1}{\ln x}\int \frac{2\ln x}{x}dx = \frac{1}{\ln x}\left((\ln x)^2 + C\right)$

$$\boxed{y = \ln x + \frac{C}{\ln x}}$$

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Ex: Solve the IVP $\begin{cases} y' + (\cot x)y = 3\cos x \sin x, \; 0<x<\pi \\ y\left(\frac{\pi}{2}\right) = 0 \end{cases}$

Sol: Integrating factor: $\mu = e^{\int \cot x\, dx} = e^{\ln|\sin x|} = |\sin x| = \sin x$

$(\sin x > 0 \text{ on } 0<x<\pi)$

Solution: $y = \frac{1}{\sin x}\int \sin x(3\cos x\sin x)dx = \frac{1}{\sin x}\int 3\sin^2 x\cos x\, dx$

$$= \frac{1}{\sin x}\left(\sin^3 x + C\right)$$

$$\boxed{y = \sin^2 x + \frac{C}{\sin x}}$$

# Mixing Problems, Revisited



Ex: Suppose we have a tank holding 100L of pure water and we start pumping in a salt water solution with concentration 2 g/L at a rate of 15 L/min. We drain the well-mixed solution at a rate of 5 L/min.

(a) Find an expression for the amount of salt in the tank for any time $t$.

(b) How much salt is in the tank when it is holding 200 L of solution?

Sol: (a) Volume in tank $= V(t) = V_0 + (r_{in} - r_{out})t = 100 + (15-5)t = 100 + 10t$

$$\begin{cases} Q'(t) = r_{in} q_{in} - r_{out} q_{out} = (15)(2) - (5)\dfrac{Q(t)}{V(t)} = 30 - \dfrac{Q(t)}{20+2t} \\ Q(0) = 0 \end{cases}$$

$$\Rightarrow Q' + \frac{Q}{20+2t} = 30 \Rightarrow \mu = e^{\int \frac{1}{20+2t} dt} = e^{\frac{1}{2}\ln|20+2t|} = \sqrt{20+2t}$$

$$Q(t) = \frac{1}{\sqrt{20+2t}} \int 30\sqrt{20+2t}\, dt = \frac{1}{\sqrt{20+2t}}\left(10(20+2t)^{3/2} + C\right) = 200 + 20t + \frac{C}{\sqrt{20+2t}}$$

$$Q(0) = 200 + 0 + \frac{C}{\sqrt{20}} \Rightarrow C = -200\sqrt{20} \Rightarrow \boxed{Q(t) = 200 + 20t - \frac{200\sqrt{20}}{\sqrt{20+2t}}}$$

(b) Tank has 200L w/ $200 = V(t) = 100 + 10t \Rightarrow t = 10$

$$Q(10) = 200 + 200 - \frac{200\sqrt{20}}{\sqrt{40}} = \boxed{400 - \frac{200}{\sqrt{2}}}$$

# Extra: Variation of Parameters

There is a slightly more general procedure one can use to solve linear 1st order ODEs. The benefit of this method is that is generalizes to higher order equations, whereas integrating factors do not. We begin by solving the homogeneous part of $y'+p(x)y=g(x)$:

$$y'+p(x)y=0 \qquad (\Delta)$$

This is always separable. We call the solution to $(\Delta)$ the homogeneous solution, and denote it by $y_H$. To get the general solution to $y'+p(x)y=g(x)$, we try looking for solutions of the form $y=uy_H$, where $u$ is some unknown function:

Plug $y=uy_H$ into $y'+p(x)y=g(x)$:

$$y'+p(x)y=u'y_H+uy_H'+p(x)uy_H=u'y_H+u\underbrace{(y_H'+p(x)y_H)}_{=0}$$

$$=u'y_H=g(x)$$

So, $u' y_H = g(x) \Rightarrow u' = \dfrac{g(x)}{y_H}$

$$\Rightarrow u = \int \frac{g(x)}{y_H}\,dx$$

Thus, the solution to $y' + p(x)y = g(x)$ is

$$\boxed{y = y_H \int \frac{g(x)}{y_H}\,dx}$$

Ex: Solve $y' + (\cot x)y = 3\cos x \sin x$, $0 < x < \pi$ using variation of parameters

check $y=0$ ✓

Sol: Homogeneous part: $y' + (\cot x)y = 0 \Rightarrow \frac{1}{y}\,dy = -\cot x\,dx$

$\Rightarrow \ln|y| = -\ln|\sin x| + C = \ln|\csc x| + C$

$\Rightarrow |y| = e^C \csc x$ ($\csc x > 0$ on $0 < x < \pi$)

$\Rightarrow y_H = C\csc x$ is homogeneous solution.

Use $y_H = \csc x$, then

$$y = \csc x \int \frac{3\cos x \sin x}{\csc x}\,dx = \csc x \int 3\sin^2 x \cos x\,dx = \csc x\left(\sin^3 x + C\right)$$

$$\Rightarrow \boxed{y = \sin^2 x + C\csc x} \left(= \sin^2 x + \frac{C}{\sin x}\right)$$